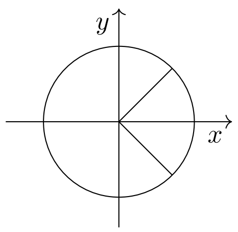
1401. Solving in $[0, 2\pi)$,

$$\begin{aligned} &\cos 3\theta = \tfrac{\sqrt{2}}{2} \\ \Longrightarrow\ &3\theta = \tfrac{\pi}{4}, \tfrac{7\pi}{4}, \tfrac{9\pi}{4}, \tfrac{15\pi}{4}, \tfrac{17\pi}{4}, \tfrac{23\pi}{4} \\ \Longrightarrow\ &\theta = \tfrac{\pi}{12}, \tfrac{7\pi}{12}, \tfrac{9\pi}{12}, \tfrac{15\pi}{12}, \tfrac{17\pi}{12}, \tfrac{23\pi}{12}. \end{aligned}$$

— Nota Bene —

The primary root is $3\theta = \pi/4$. The other values in the second line correspond to the radii shown in this unit circle, $\cos 3\theta$ being the x coordinate of the point at angle 3θ:

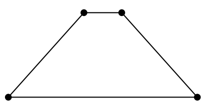


We consider three full rotations around the circle, to allow for subsequent division by 3.

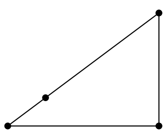
1402. Differentiating, the gradient formula is $\frac{dy}{dx} = 2x$. So, $m_{\text{tan}} = 2$ and $m_{\text{nor}} = -\frac{1}{2}$. The equation of the normal, then, is $y - 1 = -\frac{1}{2}(x - 1)$, giving $y = -\frac{1}{2}x + \frac{3}{2}$. Solving simultaneously,

$$\begin{aligned} &x^2 = -\tfrac{1}{2}x + \tfrac{3}{2} \\ \Longrightarrow\ &2x^2 + x - 3 = 0 \\ \Longrightarrow\ &(x-1)(2x+3) = 0 \\ \Longrightarrow\ &x = 1, -\tfrac{3}{2}, \text{ as required.} \end{aligned}$$

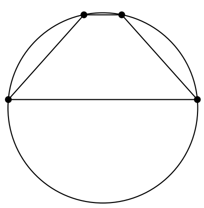
1403. (a) Yes, with sides in order $1, 3, 5, 3$:



(b) Yes, with $1 + 3$ set in a straight line to form a length 4, giving a $(3, 4, 5)$ triangle.



(c) Yes, the trapezium in part (a) is cyclic:



1404. Quoting a standard result, the derivative of $\tan\theta$ is $\sec^2\theta$. So, by the chain rule,

$$\frac{du}{d\theta} = 2\sec^2 2\theta.$$

1405. The expected scale factor is

$$\tfrac{1}{4} \times 1.1 + \tfrac{3}{4} \times 0.9 = 0.95.$$

So, the expected reduction is by 5%.

— Nota Bene —

We can only calculate the *expected* percentage change because we don't know which values in the sample are being increased and which reduced. On average, a quarter of the sum $\sum x$ will be increased and the rest reduced.

1406. Since f and g are quadratic functions, $\text{f}(x) = \text{g}(x)$ is a quadratic equation, and has a maximum of two roots. Furthermore, we know that the graphs of $y = \text{f}(x)$ and $y = \text{g}(x)$ are tangent at $x = a$, because they have the same value and the same gradient. Hence, $\text{f}(x) = \text{g}(x)$ has a *double* root at $x = a$. So, there can be no roots elsewhere. QED.

1407. Multiplying up,

$$\begin{aligned} &4 = x(x+3) \\ \Longrightarrow\ &x^2 + 3x - 4 = 0 \\ \Longrightarrow\ &x = -4, 1. \end{aligned}$$

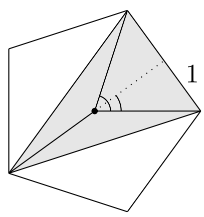
But $x = -4$ is not in the domain of the square root function, so the solution is $x = 1$.

1408. For $n \in \mathbb{N}$, the sum of three consecutive squares is

$$\begin{aligned} &(n-1)^2 + n^2 + (n+1)^2 \\ \equiv\ &3n^2 + 2. \end{aligned}$$

Since $3n^2$ is a multiple of 3 for any integer n, $3n^2+2$ cannot be. So, the sum of three consecutive squares is never a multiple of 3. □

1409. Adding three radii and a dotted perpendicular (apothem), the scenario is as shown. The apothem divides the side length in two, so the right-angled triangles have a side length of 1:



One sector subtends $\frac{2\pi}{5}$ radians at the centre, so one half-sector subtends $\frac{\pi}{5}$. Hence, using the side of length 1, the radii have length $\operatorname{cosec}\frac{\pi}{5}$, and the apothem has length $\cot\frac{\pi}{5}$. The perpendicular height of the shaded triangle is the sum of these:

$$h = \operatorname{cosec}\tfrac{\pi}{5} + \cot\tfrac{\pi}{5}.$$

Since the base is 2, this is also the area.

1410. From the formula/differentiating/completing the square, we know that $y = ax^2 + bx + c$ has a line of symmetry at $x = -\frac{b}{2a}$. The inputs $-\frac{b}{2a} - x$ and $-\frac{b}{2a} + x$ are reflections of each other in this line. Hence, they produce identical outputs. □

1411. For fixed points,

$$\begin{aligned} a &= \sqrt{12a - 36} \\ \Longrightarrow a^2 &- 12a + 36 = 0 \\ \Longrightarrow a &= 6. \end{aligned}$$

The square root is well defined at $a = 6$, so 6 is the one fixed point of the iteration.

1412. A counterexample is $\text{f}(x) = x$ and $\text{g}(x) = 10 - x$. Apply either function twice, and you get $x \mapsto x$. So, the functions f and g are not identically equal, but the functions f^2 and g^2 are.

1413. In the fourth line, there is a division by zero. When working algebraically, you shouldn't divide by an unknown without considering whether that unknown could be zero.

1414. The possibility space is

	1	2	3	4	5	6
1		✓		✓		✓
2	✓		✓		✓	
3		✓		✓		✓
4	✓		✓		✓	
5		✓		✓		✓
6	✓		✓		✓	

So, the probability is $\frac{18}{36} = \frac{1}{2}$.

1415. (a) Scaling the outputs gives $[-1, 1]$.
(b) Squaring maps $[-a, a]$ to $\left[0, a^2\right]$. Hence, the range of the function is $[0, 1]$.

1416. (a) For equilibrium, the centre of mass must lie in between the two chains. Hence, the length c to the right of the right-hand chain cannot exceed half of the length of the light: $c \leq \frac{1}{2}(a+b+c)$. Rearranging this gives the required result.
(b) The centre of mass is in the middle, since the light is uniform. So, the ratio of distances from the centre is

$$\begin{aligned} & b + c - \frac{a+b+c}{2} : a + b - \frac{a+b+c}{2} \\ &= \frac{b+c-a}{2} : \frac{a+b-c}{2} \\ &= b + c - a : a + b - c. \end{aligned}$$

Taking moments around the centre of mass, the ratio of tensions must be the reciprocal of the ratio of distances, which gives us the result.

1417. Using the fact that $\log_p q$ and $\log_q p$ are reciprocal,

$$\log_p q^2 \times \log_q p^3 = 2 \log_p q \times 3 \log_q p = 6.$$

1418. One hour is $\frac{\pi}{6}$ radians on a clock. So, at 3:20pm, the minute hand is at $\frac{4\pi}{6}$ radians from 12, and the hour hand is at $\left(3 + \frac{1}{3}\right) \times \frac{\pi}{6} = \frac{5}{9}$ radians from 12. The difference between hands is $\frac{\pi}{9}$ radians.

1419. The equation for intersections of $x^2 + y^2 = 1$ and $2x + 3y = 4$ is

$$\begin{aligned} & \left(2 - \tfrac{3}{2}y\right)^2 + y^2 = 1 \\ \Longrightarrow\ & \tfrac{13}{4}y^2 - 6y + 3 = 0. \end{aligned}$$

The discriminant is

$$\Delta = 36 - 4 \cdot \tfrac{13}{4} \cdot 3 = -3 < 0.$$

Hence, the straight line passes above the circle, and, since $2x + 3y > 4$ is satisfied by points above the line, no (x, y) points satisfy both conditions.

1420. We want the following number to be rational:

$$\begin{aligned} & \left(x^{\frac{1}{3}} + 1\right)\left(x^{\frac{2}{3}} + px^{\frac{1}{3}} + q\right) \\ \equiv\ & x + (1+p)x^{\frac{2}{3}} + (p+q)x^{\frac{1}{3}} + q. \end{aligned}$$

If x is not a perfect cube, then $x^{\frac{2}{3}}$ is irrational, so we require $(1 + p) = 0$, so $p = -1$. Likewise with $x^{\frac{1}{3}}$, so $q = 1$.

1421. Expanding the notation,

$$\begin{aligned} & \left(4a - 2a^2\right) - \left(a - a^2\right) = 0 \\ \Longrightarrow\ & 3a - 3a^2 = 0 \\ \Longrightarrow\ & a(1 - a) = 0 \\ \Longrightarrow\ & a = 0, 1. \end{aligned}$$

1422. The factor theorem states that: $(x - \alpha)$ divides exactly into $\text{f}(x)$ if and only if $\text{f}(\alpha) = 0$. And, if $A \iff B$, it must also be true that $A' \iff B'$. Hence, the implication is $\iff$.

1423. The lower bound is attained if all four angles are equal at $\frac{\pi}{2}$.

For the upper bound, the least the smallest angle can be is 0, which gives the angles as $\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi\}$. This bound is, however, not attainable.

So, the set of possible values is $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right)$ radians.

1424. (a) Differentiating, the first derivative is $\frac{dy}{dx} = 3x^2$. This gives $m = 3p^2$ at $x = p$.
(b) The equation of the tangent is $y = 3p^2x + c$, and it passes through point $\left(p, p^3\right)$. Subbing this point in, $p^3 = 3p^2 \cdot p + c$, so $c = -2p^3$. Hence, the tangent is $y = 3p^2x - 2p^3$.

1425. The statements are different, but statement (b) implies statement (a). And statement (b) is true, because every value in $[0,1]$ can be attained by $\mathrm{h}(x)$, hence every value in $[c, c+1]$ can be attained by $\mathrm{h}(x)+c$. So, statement (a) is also true.

(a) True.

(b) True.

1426. (a) There will be no change: $\sum x^2$ isn't involved in calculation of the mean.

(b) S_{xx} will increase with $\sum x^2$. The formula is

$$S_{xx} = \sum x^2 - n\bar{x}^2.$$

(c) s will increase with S_{xx}. The formula is

$$s = \sqrt{\frac{S_{xx}}{n}}.$$

1427. The first factor can be ignored, because x^2+5 is always positive. The remaining two factors have sign changes at $x=1$ and $x=-2/3$ respectively. Hence, the solution is $x \in (\infty, -2/3] \cup [1, \infty)$.

1428. (a) Subtracting "not-B" from A gives $A \cap B$.

(b) The three brackets are the three regions of the union $A' \cup B'$.

1429. The logarithms are reciprocals of each other. So,

$$\log_x y + \frac{1}{\log_x y} = 2.$$

Multiplying by $\log_x y$,

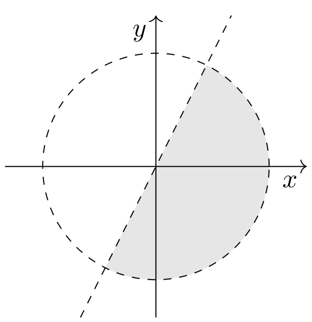
$$\begin{aligned} & (\log_x y)^2 - 2\log_x y + 1 = 0 \\ \Longrightarrow\ & (\log_x y - 1)^2 = 0 \\ \Longrightarrow\ & \log_x y = 1 \\ \Longrightarrow\ & x = y, \text{ as required.} \end{aligned}$$

1430. Consider a population consisting of

$$\underbrace{0, ..., 0}_{100}, 1, 2, 3, 4, 5, 6, 7, 8, 9, \underbrace{10, ..., 10}_{50}.$$

This is bimodal, since 0 and 10 represent distinct peaks. But the mode is 0 and the median is 0.

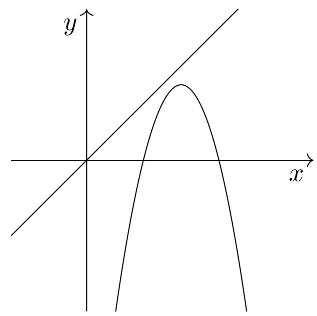
1431. The boundaries are $y = 2x$ and $x^2+y^2 = \frac{1}{4}$, which is a circle of radius $\frac{1}{2}$ centred on the origin.



1432. Equating input x and output $\mathrm{h}(x)$,

$$\begin{aligned} & 4 - (x-5)^2 = x \\ \Longrightarrow\ & x^2 - 9x + 29 = 0. \end{aligned}$$

This has discriminant $\Delta = 81 - 4 \times 29 = -35 < 0$, so the graphs $y = \mathrm{h}(x)$ and $y = x$ do not intersect. And, since $y = \mathrm{h}(x)$ is a negative parabola, it must be below $y = x$ for all x values.



Therefore, $x > \mathrm{h}(x)$ for all x.

1433. The equation for intersections is

$$x^2 + 2x + 3 = x^2 + px + q.$$

Solving this,

$$2x + 3 = px + q \implies x = \frac{q-3}{2-p}.$$

This solution is valid for all values of the constants p and q except $p = 2$, which would involve division by zero. So, $p \in \mathbb{R} \setminus \{2\}$ and $q \in \mathbb{R}$.

1434. The cosine rule produces an angle

$$\theta = \arccos \frac{10^2 + 17^2 - 21^2}{2 \cdot 10 \cdot 17}.$$

The sine area formula then gives

$$A = \tfrac{1}{2} \cdot 10 \cdot 17 \sin\theta = 84 \text{ km}^2.$$

Alternative Method

The semiperimeter is $s = \frac{1}{2}(10+17+21) = 24$. Using Heron's formula,

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{24(24-10)(24-17)(24-21)} \\ &= 84 \text{ km}^2. \end{aligned}$$

1435. Solving to find intersections, we substitute for y, which gives a quadratic in $\sqrt{x}$:

$$\begin{aligned} 2\sqrt{x}\sqrt{a} &= x + a \\ \implies x - 2\sqrt{x}\sqrt{a} + a &= 0 \\ \implies (\sqrt{x} - \sqrt{a})^2 &= 0 \\ \implies x &= a. \end{aligned}$$

So, line and curve intersect at $x = a$. Furthermore, since $\left(\sqrt{x} - \sqrt{a}\right)$ is squared, this is a double root. Hence, the point of intersection $x = a$ is a point of tangency.

—— Alternative Method ——

The gradient at $x = a$ is

$$\tfrac{1}{2}x^{-\frac{1}{2}}\Big|_{x=a} = \frac{1}{2\sqrt{a}}.$$

So, the equation of the tangent at $\left(a, \sqrt{a}\right)$ is

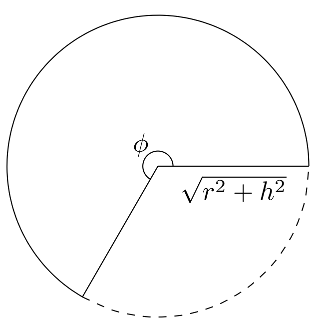
$$\begin{aligned} y - \sqrt{a} &= \frac{1}{2\sqrt{a}}(x - a) \\ \implies 2y\sqrt{a} - 2a &= x - a \\ \implies 2y\sqrt{a} &= x + a, \text{ as required.} \end{aligned}$$

1436. (a) Beam a and its symmetrical counterpart must be in tension, because they are the only beams which can exert a vertical force on the load at P. Since gravity is pulling P down, there must be tension in beam a pulling P up.

(b) If beam b were to disappear, there would be nothing stopping the upper two joints moving towards each other, and hence the load falling. So, beam b must be in compression, holding the upper joints apart.

1437. The outputs of function g must be throughput into function f. Hence, every element of set D must also be an element of set A, the permitted inputs of f. So, the necessary relationship is $D \subset A$.

1438. Pythagoras gives the slant height of the cone as $l = \sqrt{r^2 + h^2}$. This becomes the sector radius when the cone is unwrapped:



The major arc length is the base circumference of the cone, which is $2\pi r$. The full circumference is $2\pi\sqrt{r^2 + h^2}$. So, this gives us a ratio of angles

$$\begin{aligned} \frac{\phi}{2\pi} &= \frac{2\pi r}{2\pi\sqrt{r^2 + h^2}} \\ \implies \phi &= \frac{2\pi r}{\sqrt{r^2 + h^2}}, \text{ as required.} \end{aligned}$$

1439. Let $\text{F}(x)$ be the integral of y. Then the LHS is

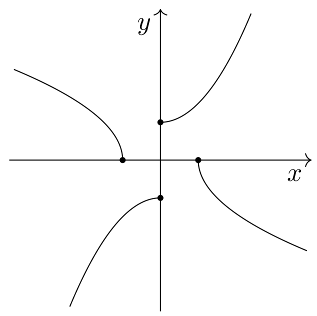
$$\begin{aligned} &\Big[\text{F}(x)\Big]_a^c + \Big[\text{F}(x)\Big]_b^d \\ &= \text{F}(c) - \text{F}(a) + \text{F}(d) - \text{F}(b). \end{aligned}$$

Similarly, the RHS is

$$\begin{aligned} &\Big[\text{F}(x)\Big]_a^d + \Big[\text{F}(x)\Big]_b^c \\ &= \text{F}(d) - \text{F}(a) + \text{F}(c) - \text{F}(b). \end{aligned}$$

Rearranging the terms, the LHS and RHS are equal, which proves the result. □

1440. (a) The pattern is



(b) In each case, we can transform one of the given curves by rotating 180° around the origin, which involves replacing (x, y) with $(-x, -y)$. The equations are $y = \sqrt{-x - 1}$ in the second quadrant and $y = -x^2 - 1$ in the third.

1441. We multiply top and bottom of the main fraction by $2x$, and then square both sides of the equation:

$$\begin{aligned} \frac{1 - \frac{1}{2x}}{\sqrt{1 - \frac{1}{4x^2}}} &= 2 \\ \implies \frac{2x - 1}{\sqrt{4x^2 - 1}} &= 2 \\ \implies \frac{(2x-1)^2}{4x^2 - 1} &= 4. \end{aligned}$$

Multiplying by the denominator,

$$\begin{aligned} (2x - 1)^2 &= 16x^2 - 4 \\ \implies 4x^2 - 4x + 1 &= 16x^2 - 4 \\ \implies 12x^2 + 4x - 5 &= 0 \\ \implies x &= \tfrac{1}{2}, -\tfrac{5}{6}. \end{aligned}$$

The original LHS is undefined at $x = \frac{1}{2}$, however. So, the solution is $x = -\frac{5}{6}$.

1442. The graph shown has a triple root at $x = 0$ and a single root at some positive $x = \alpha$, meaning that its equation must have a triple factor of x and a single factor of $(x - \alpha)$. Only equation ③ has this.

(a) No.

(b) No.

(c) Yes.

1443. Multiplying out, the terms in x cancel:

$$\begin{aligned}&\int_0^1 15\big(x+\sqrt{x}\big)\big(1-\sqrt{x}\big)\,dx\\ =\ &15\int_0^1 \sqrt{x} - x\sqrt{x}\,dx.\end{aligned}$$

Using an index law, this is

$$\begin{aligned}&15\int_0^1 x^{\frac{1}{2}} - x^{\frac{3}{2}}\,dx\\ =\ &15\Big[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\Big]_0^1\\ =\ &15\big(\tfrac{2}{3}-\tfrac{2}{5}\big) - 15\big(0\big)\\ =\ &4, \text{ as required.}\end{aligned}$$

1444. This is true. The object is in equilibrium under the action of three forces, so these forces must form a closed triangle. And, since the magnitudes satisfy

$$\big(\tfrac{3}{5}Q\big)^2 + \big(\tfrac{4}{5}Q\big)^2 \equiv Q^2,$$

this triangle of forces must be right-angled. Hence, the smaller two forces are perpendicular. □

1445. Setting to zero,

$$\begin{aligned}x^2+6x-8&=0\\ \Longrightarrow x&=\frac{-6\pm\sqrt{68}}{2}\\ &=-3\pm\sqrt{17}.\end{aligned}$$

So, according to the factor theorem, the original quadratic expression has factors $\big(x+3\mp\sqrt{17}\big)$. The quadratic is monic, so factorises as

$$x^2+6x-8\equiv\big(x+3-\sqrt{17}\big)\big(x+3+\sqrt{17}\big).$$

1446. There are two successful outcomes. If 5 is rolled, then there is a $0.25\times\frac{1}{2}$ probability that the croupier will call 6; if 6 is rolled, then there is a 0.75 probability that the croupier will call 6. So, conditioning on the true score, the probability is

$$\begin{aligned}&\mathbb{P}(\text{6 called})\\ =\ &\mathbb{P}(\text{6 called} \mid \text{5 true}) + \mathbb{P}(\text{6 called} \mid \text{6 true})\\ =\ &\tfrac{1}{6}\times 0.25\times\tfrac{1}{2}+\tfrac{1}{6}\times 0.75\\ =\ &\tfrac{7}{48}, \text{ as required.}\end{aligned}$$

1447. Considering the negation of ①, for $n \in \mathbb{Z}$, we need $n \in [1, 2, ...)$. Negating ②, we can then rule 3 and 4 out. This leaves $n \in [1, 2, 5, 6, 7, ...)$. Statement ③ then rules out $5, 6, 7....$ So, to satisfy none of the three statements, $n \in \{1, 2\}$.

1448. (a) Splitting the sum up,

$$\begin{aligned}&\sum_{i=1}^{n}(u_i+1)\\ \equiv\ &\sum_{i=1}^{n}u_i+\sum_{i=1}^{n}1\\ \equiv\ &\mathrm{S}(n)+n.\end{aligned}$$

(b) Here, we take a factor of two out, and use the standard sum of the first n integers:

$$\begin{aligned}&\sum_{i=1}^{n}(2u_i+i)\\ \equiv\ &2\sum_{i=1}^{n}u_i+\sum_{i=1}^{n}i\\ \equiv\ &2\,\mathrm{S}(n)+\tfrac{1}{2}n(n+1).\end{aligned}$$

1449. Collision occurs if $x_1 = x_2$. This gives

$$\begin{aligned}\frac{1}{1-t}&=\frac{2}{1-2t}\\ \Longrightarrow 1-2t&=2-2t\\ \Longrightarrow 1&=2.\end{aligned}$$

This is impossible. So, the particles do not collide.

1450. (a) Assume, for a contradiction, that $\log_2 3 \in \mathbb{Q}$. We can therefore write $\log_2 3$ as $\frac{a}{b}$, where $a, b \in \mathbb{N}$:

$$\begin{aligned}&\log_2 3 = \tfrac{a}{b}\\ \Longrightarrow\ &2^{\frac{a}{b}} = 3\\ \Longrightarrow\ &2^a = 3^b \text{ (raising to the power } b).\end{aligned}$$

(b) The only prime factor of the LHS 2^a is 2, so it can have no factors of 3. Hence, neither does 3^b. This means $b = 0$, which is a contradiction. Therefore, $\log_2 3$ is irrational. □

1451. The horizontal diameter is a line of symmetry of the shape. Furthermore, in each symmetrical pair of triangles, exactly one is shaded. So, half of the decagon is shaded, as required.

1452. For stationary values of z,

$$\begin{aligned}&\tfrac{d}{dx}(x^5-2x^4+x^3)=0\\ \Longrightarrow\ &5x^4-8x^3+3x^2=0\\ \Longrightarrow\ &x^2(5x^2-8x+3)=0\\ \Longrightarrow\ &x^2(5x-3)(x-1)=0\\ \Longrightarrow\ &x\in\{0, 3/5, 1\}.\end{aligned}$$

Nota Bene

"z is stationary with respect to x" means broadly that, as x changes, z does not change. But this statement needs to be made precise. It is only *instantaneously* true, i.e. only true in a limiting sense (zooming in infinitely far). It is true is the same sense as, when a ball is thrown vertically up in the air, the ball is instantaneously stationary at the top of its path. It is stationary for no length of time.

1453. Integrating $\mathrm{g}'(x) = a$, we have a generic linear function $\mathrm{g}(x) = ax+b$. The mean of the set $\{\mathrm{g}(x_i)\}$ is given, then, by

$$\begin{aligned}&\tfrac{1}{n}\sum_{i=1}^{n}\mathrm{g}(x_i)\\ =\ &\tfrac{1}{n}\sum_{i=1}^{n}a(x_i)+b.\end{aligned}$$

Splitting the sum up, and then taking out a factor of a, this is

$$\begin{aligned}&a\tfrac{1}{n}\sum_{i=1}^{n}(x_i)+\tfrac{1}{n}\sum_{i=1}^{n}(x_i)b\\ &= a\bar{x}+b\\ &= \mathrm{g}(\bar{x}),\ \text{as required.}\end{aligned}$$

1454. Putting the fractions over a common denominator,

$$\frac{36}{323}=\frac{a+b}{ab}.$$

The LHS is in its lowest terms. The RHS is also in its lowest terms, because a and b are primes: the only prime factors of the denominator are a and b, and these cannot be factors of the numerator. Hence, we can equate the numerators and equate the denominators, giving $36 = a+b$ and $323 = ab$. Solving these,

$$\frac{36}{323}=\frac{1}{17}+\frac{1}{19}.$$

1455. (a) i. Since 245 is the mean, $\mathbb{P}(M > 245) = \frac{1}{2}$. So the probability is $(1/2)^2 = 0.25$.

ii. Using a calculator cumulative distribution function, we find $\mathbb{P}(M < 250) = 0.516618$. Squaring this gives the required probability as 0.267 (3sf).

(b) Firstly, there is no reason to think that masses can be modelled with a normal distribution at all: the normal distribution applies in specific circumstances, and very rarely for individuals in biology.

Secondly, since the rodents were caught in the same trap, the two trials (selection of rodents) are certainly dependent, which invalidates the multiplication of probabilities.

1456. Using the identity $\tan x \equiv \frac{\sin x}{\cos x}$,

$$\begin{aligned}&\sin\left(2x+\tfrac{\pi}{3}\right)=\cos\left(2x+\tfrac{\pi}{3}\right)\\ \Longrightarrow\ &\tan\left(2x+\tfrac{\pi}{3}\right)=1\\ \Longrightarrow\ &2x+\tfrac{\pi}{3}=\tfrac{5\pi}{4},\tfrac{9\pi}{4},\tfrac{13\pi}{4},\tfrac{17\pi}{4}\\ \Longrightarrow\ &x=\tfrac{11\pi}{24},\tfrac{23\pi}{24},\tfrac{35\pi}{24},\tfrac{47\pi}{24}\end{aligned}$$

Nota Bene

There are infinitely many roots to this equation. The four values in line 3 were selected as follows:

- we begin with values beyond the primary $\pi/4$, because of the subsequent subtraction of $\pi/3$, which would render the primary value −ve,
- we write down the next **four** values, because of subsequent division by 2, which will double the number of roots in one period from two to four.

1457. Friction is indeed generated, which manifests as a Newton III pair of forces. But the student has the directions wrong. The two forces generated are a frictional force backwards on the ground, and a frictional force of equal magnitude forwards on the bicycle.

1458. (a) Differentiating,

$$\begin{aligned}&y=x^4-x^3\\ \Longrightarrow\ &\frac{dy}{dx}=4x^3-3x^2\\ \Longrightarrow\ &\frac{d^2y}{dx^2}=12x^2-6x.\end{aligned}$$

(b) The second derivative is $6x(2x-1)$. This has a single root at $x = 0$, so changes sign at the origin.

(c) The second derivative changes sign at $x = 0$, so the curve is neither concave nor convex on any interval surrounding the origin.

1459. Setting the numerator to zero, $x = a$ or $x = -b$. And, since a, b, c, d are distinct positive constants, the numerator and denominator share no common factors. So, the roots of the equation are the roots of the numerator: $x = a$, $x = -b$.

1460. Substituting a generic AP term $a_n = a + (n-1)d$ into the formula for b_n,

$$\begin{aligned}b_n&=10^{a_n}\\ &=10^{a+(n-1)d}\\ &\equiv 10^a\times\left(10^d\right)^{n-1}.\end{aligned}$$

This is the standard ordinal formula $u_n = ar^{n-1}$ of a geometric progression, with first term 10^a and common ratio 10^d.

1461. The first event is twice as likely:

$$\mathbb{P}(\text{ace, king, queen}) = 3! \times \left(\tfrac{1}{13}\right)^3$$
$$\mathbb{P}(4{,}4{,}5) = {}^3\text{C}_1 \times \left(\tfrac{1}{13}\right)^3.$$

1462. (a) By Pythagoras, $|\mathbf{p}| = |\mathbf{q}| = 5$.

(b) We solve simultaneously by elimination:

$$3\mathbf{p} = 9\mathbf{i} + 12\mathbf{j}$$
$$4\mathbf{q} = 16\mathbf{i} - 12\mathbf{j}$$

Adding gives $3\mathbf{p} + 4\mathbf{q} = 25\mathbf{i}$. Hence,

$$\mathbf{i} = \tfrac{1}{25}(3\mathbf{p} + 4\mathbf{q})$$
$$\mathbf{j} = \tfrac{1}{25}(4\mathbf{p} - 3\mathbf{q}).$$

1463. Carrying out the integrals,

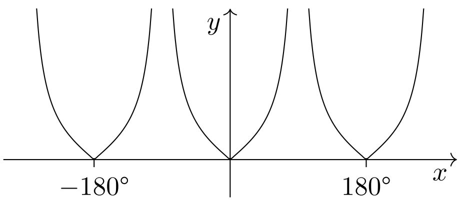
$$\tfrac{1}{2}x^2 + c_1 + \tfrac{1}{2}y^2 + c_2 = 0.$$

Combining the constants and multiplying by two, this can be expressed as $x^2 + y^2 = c$, which is the equation of a family of circles of varying radius, centred on the origin.

1464. Each exam is scaled to total 50 marks by the scale factors $\frac{50}{A}$, $\frac{50}{B}$. Then, adding gives the formula:

$$M = \frac{50a}{A} + \frac{50b}{B}.$$

1465. Applying the modulus function to the outputs of the tan function takes all (x, y) points on the graph $y = \tan x$ for which $y < 0$ and reflects them in the x axis, giving



1466. Both equations can be written in terms of z and $x - y$, which we call a:

$$a + z = 3,$$
$$2a - z = 0.$$

Solving, we find $a = 1, z = 2$.

— Nota Bene —

In general, such a problem involving two equations and three unknowns would not be possible to solve. And, indeed, it is not possible to solve for x and y. It is only because x and y appear in the same combination $x - y$ in both equations that a value for z (and for $x - y$) can be found.

1467. Multiplying out gives $pqx^2 + (p+q)x + 1 = 0$. So, the discriminant Δ is

$$\begin{aligned} &(p+q)^2 - 4pq \\ &\equiv p^2 + 2pq + q^2 - 4pq \\ &\equiv p^2 - 2pq + q^2 \\ &\equiv (p-q)^2, \text{ as required.} \end{aligned}$$

1468. (a) The top two branches have total probability $\frac{4}{20} = \frac{1}{5}$, which is the value of a. Using this,

$$\begin{aligned} &\mathbb{P}(X' \cap Y) = \tfrac{4}{20} \\ \Longrightarrow\ &\tfrac{4}{5}b = \tfrac{4}{20} \\ \Longrightarrow\ &b = \tfrac{1}{4}. \end{aligned}$$

(b) $\mathbb{P}(X \cup Y') = 1 - \mathbb{P}(X' \cap Y) = \frac{16}{20} = \frac{4}{5}$.

1469. Multiplying by 100 and writing as $\text{f}(x) = 0$, the Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{4x_n^3 - 7x_n^2 - 62x_n - 15}{12x_n^2 - 14x_n - 62}.$$

Running the iteration with $x_0 = 0$ gives $x_n \to -\frac{1}{4}$. So, by the factor theorem, we know $(4x + 1)$ is a factor. We can factorise, therefore, as

$$\begin{aligned} &4x^3 - 7x^2 - 62x - 15 = 0 \\ \Longrightarrow\ &(4x+1)(x^2 - 2x - 15) = 0 \\ \Longrightarrow\ &(4x+1)(x+3)(x-5) = 0 \\ \Longrightarrow\ &x = -3, -\tfrac{1}{4}, 5. \end{aligned}$$

— Alternative Method —

The polynomial long division is

$$\begin{array}{r|l} & \quad\ \ x^2 \quad - 2x - 15 \\ \hline 4x+1 & 4x^3 - 7x^2 - 62x - 15 \\ & -\ 4x^3 \ \ - x^2 \\ \hline & \qquad - 8x^2 - 62x \\ & \qquad\ \ \ 8x^2 \ + 2x \\ \hline & \qquad\qquad - 60x - 15 \\ & \qquad\qquad\ \ \ 60x + 15 \\ \hline & \qquad\qquad\qquad\quad 0 \end{array}$$

1470. Since the integral of e^x is e^x, by the (reverse) chain rule, we require an extra minus sign:

$$\int e^x + e^{-x}\,dx = e^x - e^{-x} + c.$$

1471. Solving for intersections,

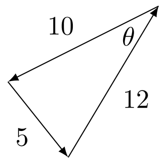
$$\begin{aligned} &9 - 2(x-3)^2 = 2x^2 \\ \Longrightarrow\ &4x^2 - 12x + 9 = 0. \end{aligned}$$

The discriminant is $\Delta = 144 - 4 \times 4 \times 9 = 0$. So, this equation has a double root. Hence, the point of intersection must be a point of tangency.

1472. Since $[0, 1]$ is a continuous set, the probability of equality is zero. So, there are only two directions in which the inequality can go. These are equally likely, since the pairs of values $x_1 + x_2$ and $x_3 + x_4$ are symmetrical. Hence, $p = \frac{1}{2}$.

1473. This is not true. A counterexample is $A : x = 1$, $B : x^2 = 1$ and $C : x = -1$.

1474. Setting up the triangle of forces, we have



The cosine rule gives

$$\theta = \arccos \frac{10^2 + 12^2 - 5^2}{2 \cdot 10 \cdot 12} = 24.146...^\circ.$$

We want the obtuse angle, so $\theta = 155.9$ (1dp).

1475. We can eliminate both x and z by subtracting equation 3 from equation 1. This gives $2y = -2$, so $y = -1$. The first two equations are then

$$\begin{aligned} x + z &= 1 \\ x - z &= 2. \end{aligned}$$

Solving these gives (x, y, z) as $(3/2, -1, -1/2)$.

1476. Splitting the sum up,

$$\begin{aligned} &\sum_{i=1}^{n} (x_i - \bar{x}) \\ = &\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \bar{x}. \end{aligned}$$

The first sum is equal to $n\bar{x}$ by definition of the mean $\bar{x}$. The second sum is equal to $n\bar{x}$ because it consists of n copies of $\bar{x}$. The difference, therefore, is zero. □

1477. (a) We know that $\mathrm{f}(x) = 0$ has exactly one real root, and that $\mathrm{g}(x) = 0$ has exactly two. The root of $\mathrm{f}(x) = 0$ may also be a root of $\mathrm{g}(x) = 0$, or all three may be distinct. So, there are two possibilities: two or three real roots.

(b) The square doesn't change the argument: the equation has two or three real roots.

(c) Likewise, two or three real roots.

— Nota Bene —

The powers α and β on the factors $\mathrm{f}(x)$ and $\mathrm{g}(x)$ in $\mathrm{f}(x)^\alpha\, \mathrm{g}(x)^\beta = 0$ do not affect the possible numbers of roots: z^α is zero if and only if z is zero.

1478. The octahedron is symmetrical, so we can pick the first face without loss of generality. There are then seven faces remaining, of which three share an edge with the first. So, the probability is $\frac{3}{7}$.

1479. Using the binomial expansion,

$$(e^x \pm 1)^4 \equiv e^{4x} \pm 4e^{3x} + 6e^{2x} \pm 4e^x + 1.$$

When we subtract, the even terms cancel, leaving

$$(e^x + 1)^4 - (e^x - 1)^4 \equiv 8e^{3x} + 8e^x.$$

1480. (a) Multiplying for elimination,

$$\begin{aligned} adx + bdy &= dp, \\ bcx + bdy &= bq. \end{aligned}$$

Subtracting these,

$$\begin{aligned} adx - bcx &= dp - bq \\ \Longrightarrow x &= \frac{dp - bq}{ad - bc}. \end{aligned}$$

Substituting back in and simplifying,

$$y = \frac{cp - aq}{ad - bc}.$$

(b) If $ad \neq bc$, then $ad - bc \neq 0$.

- In algebraic terms, the condition is there to avoid division by zero.
- In geometric terms, the condition is there is guarantee that the two straight lines aren't parallel, i.e. that they will intersect at a single (x, y) point.

1481. We need to show that the denominator of the LHS and the RHS multiply to give 1. Writing as powers of x, we have

$$\begin{aligned} &\left(1 + 2^{\frac{1}{4}}\right)\left(2^{\frac{1}{4}} - 2^{\frac{1}{2}} + 2^{\frac{3}{4}} - 1\right) \\ = &\left(2^{\frac{1}{4}} - 2^{\frac{1}{2}} + 2^{\frac{3}{4}} - 1\right) + \left(2^{\frac{1}{2}} - 2^{\frac{3}{4}} + 2 - 2^{\frac{1}{4}}\right) \\ = &-1 + 2 \\ = &\ 1, \text{ as required.} \end{aligned}$$

1482. (a) Replace y by $-y$, giving $-y = ax^2 + bx + c$, or equivalently $y = -ax^2 - bx - c$.

(b) Replace x by $-x$, giving $y = ax^2 - bx + c$,

(c) Switch x and y, giving $x = ay^2 + by + c$.

1483. Taking out common factors,

$$\begin{aligned} &(x + 2)^3(x - 1) + (x + 2)(x - 1)^3 = 0 \\ \Longrightarrow\ &(x + 2)(x - 1)\left[(x + 2)^2 + (x - 1)^2\right] = 0 \\ \Longrightarrow\ &(x + 2)(x - 1)\left[2x^2 + 2x + 5\right] = 0. \end{aligned}$$

The discriminant of the quadratic factor is

$$\Delta = 2^2 - 4 \cdot 2 \cdot 5 = -36 < 0.$$

So, the solution is $x = -2, 1$.

1484. The information given restricts the 36 outcomes of the possibility space as follows:

	1	2	3	4	5	6
1					✓	✓
2				✓	✓	✓
3			✓	✓	✓	✓
4		✓	✓	✓	✓	✓
5	✓	✓	✓	✓	✓	✓
6	✓	✓	✓	✓	✓	✓

There are 15 successful outcomes in the restricted possibility space, so the probability is $\frac{15}{25} = \frac{3}{5}$.

1485. Using index laws, we have

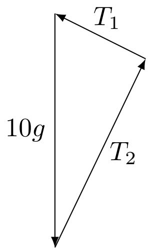
$$\begin{aligned} a &= 2^{x+1} \\ &\equiv 2 \times 2^x \\ &\equiv 2 \times \sqrt{4^x} \\ &\equiv 2 \times 2 \times \sqrt{4^x \div 4} \\ &= 4\sqrt{b}. \end{aligned}$$

1486. Using a polynomial solver on the calculator, the cubic has roots at $x = -2, -\frac{1}{6}, \frac{1}{3}$. Using these, we can reconstruct an algebraic solution:

$$\begin{aligned} & 18x^3 + 33x^2 - 7x - 2 = 0 \\ \Longrightarrow\ & (x+2)(6x+1)(3x-1) = 0 \\ \Longrightarrow\ & x = -2, -\tfrac{1}{6}, \tfrac{1}{3}. \end{aligned}$$

1487. (a) The cables form a Pythagorean triple. So, the angles of inclination (from the horizontal) are given by right-angled trig: $\arccos \frac{24}{25} = 16.26°$ and $\arcsin \frac{24}{25} = 73.74°$.

(b) The triangle of forces, modelling the tensions and the weight, is as follows:



This is right-angled too, with sides in the ratio $7 : 24 : 25$. Therefore, $T_1 = \frac{70}{25}g = 2.8g$ and $T_2 = \frac{240}{25}g = 9.6g$. The tensions are 27.44 N and 94.08 N.

— Nota Bene —

Note that, in the triangle of forces, the *longer* side corresponds to the *shorter* string.

1488. Setting $\arcsin x = y$, we have $x = \sin y$. So,

$$\begin{aligned} & \sqrt{1 - x^2} \\ =\ & \sqrt{1 - \sin^2 y} \\ =\ & \sqrt{\cos^2 y} \\ =\ & \cos y \\ =\ & \cos(\arcsin x). \end{aligned}$$

1489. Differentiating implicitly, we treat the term y^2 as a composition of two functions: the inside function is $x \mapsto y$, the outside function is "squaring". The chain rule, then, gives

$$\begin{aligned} & \frac{d}{dx}(x^2 + y^2) = 0 \\ \Longrightarrow\ & 2x + 2y\frac{dy}{dx} = 0 \\ \Longrightarrow\ & \frac{dy}{dx} = -\frac{x}{y}. \end{aligned}$$

— Nota Bene —

If such implicit differentiation is hard to visualise, replace y with $\mathrm{f}(x)$ everywhere in both question and answer. Then run the logic again.

1490. Since the fraction may be written as a quadratic, the denominator must divide into the numerator. Hence, by the factor theorem, $x = -\frac{1}{2}$ is a root of the numerator, giving $-\frac{1}{2}b - \frac{7}{2} = 0$. So, $b = -7$.

1491. (a) With a set of the low-valued data removed, the sample mean will increase.

(b) With five extreme-valued data removed, the sample standard deviation will decrease.

1492. We need the perpendicular bisector of AB. Since $m_{AB} = -\frac{1}{2}$ and the midpoint of AB is at $(3, 2)$, this has equation $y = 2x - 4$.

1493. Multiplying up by the denominators,

$$\begin{aligned} & (x+1)(x+2) + x(x+2) + x(x+1) = 0 \\ \Longrightarrow\ & x^2 + 3x + 2 + x^2 + 2x + x^2 + x = 0 \\ \Longrightarrow\ & 3x^2 + 6x + 2 = 0. \end{aligned}$$

This is a quadratic with discriminant $\Delta = 12 > 0$, so it has two distinct real roots, as required.

1494. Simplifying the summand,

$$\log_8 \left(2^i\right) \equiv i \log_8 2 \equiv \tfrac{1}{3}i.$$

Hence, we have an arithmetic series with first term $a = \frac{1}{3}$, common difference $d = \frac{1}{3}$ and $n = 50$ terms. The standard partial sum formula gives

$$S_{50} = \tfrac{50}{2}\left(\tfrac{2}{3} + 49 \cdot \tfrac{1}{3}\right) = 425.$$

1495. This is an identity, so we can equate coefficients. This gives $b = 3$ immediately. Considering the x term, we get $3a^2b = 225$, so (positive) $a = 5$. We can then perform the binomial expansion, giving $c = 125$ and $d = 3ab^2 = 135$.

1496. Since $(x - p + 1) \equiv (x + 1 - p)$, the transformation has replaced x with $x + 1$. This is a translation by vector $\binom{-1}{0}$.

1497. Each can be considered as an arithmetic series:

(a) $a = 2$, $d = 2$, with n terms:

$$S_n = \tfrac{n}{2}\big(4 + (n-1)2\big) \equiv n(n+1).$$

(b) $a = 1$, $d = 2$, with n terms:

$$S_n = \tfrac{n}{2}\big(2 + (n-1)2\big) \equiv n^2.$$

— Alternative Method —

We can use the standard formula for the sum of the first n integers $\sum i = \frac{1}{2}n(n+1)$.

(a) $\displaystyle\sum_1^n 2i \equiv 2\sum_1^n i \equiv n(n+1)$.

(b) $\displaystyle\sum_1^n (2i-1) \equiv 2\sum_1^n i - n \equiv n(n+1) - n \equiv n^2$.

1498. (a) $y = m_1 x$ intersects the rectangle at $(4, 4m_1)$, creating a triangle of area $\frac{1}{2} \cdot 4 \cdot 4m_1 = 8m_1$. The area of the rectangle is 12, so we require $8m_1 = 4$. Hence, $m_1 = \frac{1}{2}$, as required.

(b) Line $y = m_2 x$ intersects the rectangle at $(3/m_2, 3)$, creating a triangle of area

$$\tfrac{1}{2} \cdot \tfrac{3}{m_2} \cdot 3 = \tfrac{9}{2m_2}.$$

As before, we need $\frac{9}{2m_2} = 4$. So, $m_2 = \frac{9}{8}$.

1499. Multiplying probabilities roll by roll,

$$\mathbb{P}(\text{all different}) = 1 \times \tfrac{5}{6} \times \tfrac{4}{6} \times \tfrac{3}{6} \times \tfrac{2}{6} \times \tfrac{1}{6} = \tfrac{5}{324}.$$

— Alternative Method —

The possibility space consists of 6^6 equally likely outcomes. There are $6!$ successful outcomes, giving

$$\mathbb{P}(\text{all different}) = \frac{6!}{6^6} = \tfrac{5}{324}.$$

1500. The interior angle of a regular heptagon is given, in degrees, by $180° - \frac{360°}{7} = \frac{900°}{7}$. Dividing 360° by this value gives 2.8. Since this is not an integer, regular heptagons do not tessellate. □

— End of 15th Hundred —